Applications of the PS1 function $\psi(x)$ and the Dirichlet L-function to the $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \ln \tan x dx$

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Abstract

Applications of the PS1 function $\psi(x)$ and the Dirichlet L-function to the $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \ln \tan x dx = \frac{\pi}{4}\psi(1) + L'(1)$

Key Words : 1.Euler's constant
2. Γ -function
3.PS1 function
4.Dirichlet L-function
5.Weierstrass's Formula

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1. Introduction

To solve the integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \ln \tan x dx$

is very difficult and obscure. So I decided to study the integral in some depth it turns out that this PS1 Function and L-function to prove.

2.We will show that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \ln \tan x dx = \frac{d}{ds}\Gamma(S)L(S) \mid s = 1 \qquad (1)$$

Invoking the well-known formulas

$\Gamma(1) = 1$

$\psi(1) = -\gamma$

where γ is Euler's constant,

$$\gamma = -\psi(1) = -0.577215664 \ldots\ldots \qquad (2)$$

Equation(1) becomes

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \ln \tan x dx = \frac{\pi}{4}\psi(1) + L'(1) \qquad (3)$$

So the proof of equation (3) will consist of three parts:

a) to find $\psi(1)$

b) establishing (1) by expressing $L'(1), \psi(1)$

c) the Dirichlet L-function

3. Proof of equation (2), the PS1 function $\frac{d}{dx}\ln\Gamma(x)$, our

starting point is the reciprocal of Weierstrass's formula for $\Gamma(x)$ which is

$$\Gamma(x) = \frac{1}{x} e^{-\gamma x} \underset{n=1}{\overset{\infty}{\pi}} \frac{e^{\frac{x}{n}}}{1+\frac{x}{n}} \qquad (4)$$

Where γ is Euler's constant, Taking logarithms gives

$$\ln\Gamma(x) = -\ln x - \gamma x + \ln \underset{n=1}{\overset{\infty}{\pi}} \frac{e^{x/n}}{1+\frac{x}{n}}$$

$$= -\ln x - \gamma x - \sum_{n=1}^{\infty}[\ln(1+\frac{x}{n}) - \frac{x}{n}] \tag{5}$$

The series on the right above equation is uniformly convergent, permitting term by term differentiation

$$\frac{d}{dx}\ln\Gamma(x) = \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\frac{1}{x} - \gamma + \sum_{n=1}^{10}\frac{x}{n(x+n)} \tag{6}$$

For x=1, the series in (6) has the value 1.

So that $\psi(1) = -1 - r + \sum_{n=1}^{\infty}\frac{1}{n(1+n)} = -\gamma = -0.5772156649\cdots$ (7)

4.Proof of equation (1).

We begin with a general Dirichlet series

$$L(s) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s} \tag{8}$$

Which $f(n)$ is of polynomial growth, will converge absolutely in a half-plane Re(s)>C.

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt = \int_0^{\infty} e^{-nu}(nu)^{s-1} d(nu)$$

$$= n^s \int_0^{\infty} e^{-nu} u^{s-1} du \tag{9}$$

$$\frac{\Gamma(s)}{n^s} = \int_0^{\infty} e^{-nt} t^{s-1} dt \tag{10}$$

Hence, by absolute convergence, one gets that for Re(s)>C

$$\Gamma(s)L(s) = \Gamma(s)\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = n^s \int_0^{\infty} e^{-nt} t^{s-1} dt \sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$

$$= \sum_{n=1}^{\infty} f(n) \int_0^{\infty} e^{-nt} t^{s-1} dt$$

$$= \int_0^{\infty} (\sum_{n=1}^{\infty} f(n) e^{-nt}) t^{s-1} dt \tag{11}$$

Now let $z=e^{-t}, z^n=e^{-nt}, t=\ln\frac{1}{z}, dt=-\frac{dz}{z}$ to (11)

We have

$$\Gamma(s)L(s)=\int_0^1\left(\sum_{n=1}^{\infty}f(n)z^n\right)\left(\ln\frac{1}{z}\right)^{s-1}\frac{dz}{z} \tag{12}$$

Now let us add the restriction that f(n) be a periodic function. So there is a positive integer q such that f(n+q) = f(n) for all n and assume that f(q) = 0.
With these assumptions we have that for |z|<1

$$\sum_{n=1}^{\infty}f(n)z^n\sum_{m=0}^{\infty}\sum_{n=1}^{q-1}f(mq+n)z^{mq+n}$$

$$=\frac{\sum_{n=1}^{q-1}f(n)z^n}{1-z^q}=\frac{g(z,f)}{1-z^q} \tag{13}$$

where $g(z,f)=\sum_{n=1}^{q-1}f(n)z^n$ (14)

(14) to (12) we have

$$L(s)\Gamma(s)=\int_0^1\frac{g(z,f)(\ln\frac{1}{z})^{s-1}}{1-z^q}\frac{dz}{z} \tag{15}$$

Differentiating equation (14) by Leibniz's rule gives

$$\frac{d}{ds}L(s)\Gamma(s)=\int_0^1\frac{g(z,f)}{1-z^q}\left(\ln\frac{1}{z}\right)^{s-1}\ln\ln\left(\frac{1}{z}\right)\frac{dz}{z} \tag{16}$$

Form (16) we have

$$L'(s)\Gamma(s)+L(s)\Gamma'(s)=\int_0^1\frac{g(z,f)}{1-z^q}\left(\ln\left(\frac{1}{z}\right)\right)^{s-1}\ln\ln\left(\frac{1}{z}\right)\frac{dz}{z} \tag{17}$$

Now if L(s) converges absolutely at s = 1 to (17). This will yield

$$L'(1)\Gamma(1)+L(1)\Gamma'(1)=\int_0^1 g(z,f)\ln\ln\left(\frac{1}{z}\right)\frac{dz}{z} \tag{18}$$

To Prove (3) we let q = 4 and pick f(n) to be quadratic character (mode 4) that is

$$x_4(n)=\begin{cases}0 & \text{if} \quad n=0 \quad (\text{mod}4)\\ 1 & \text{if} \quad n=1 \quad (\text{mod}4)\\ 0 & \text{if} \quad n=2 \quad (\text{mod}4)\\ 1 & \text{if} \quad n=3 \quad (\text{mod}4)\end{cases}$$

x_4 is call the quadratic character (mod 4) because (n,4) = 1

$$x_4(n) = \begin{cases} 1 & \text{if} \exists x, s, t, x^2 = n(\text{mod}\,4) \\ -1 & \text{otherwise} \end{cases}$$

While $x_4(n) = 0$ if (n,q)>1

So we have

$q = 4,\ q(z,x) = z - z^3$ to (18)

and equation (18) becomes

$$L'(1) - rL(1) = \int_0^1 \frac{(z - z^3)\ln\ln\frac{1}{z}}{1 - z^4}\frac{dz}{z}$$

$$= \int_0^1 \ln\ln(\frac{1}{z})\frac{dz}{1+z^2} \tag{19}$$

Now let $\frac{1}{z} = \tan x, \cot x = z, dz = -\csc^2 x dx$ to (19)

We have

$$L'(1) - \gamma L(1) = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \ln\ln\tan x \frac{(-\csc^2 x)dx}{1+\cot^2 x}$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \ln\ln\tan xdx \tag{20}$$

5. $L(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$ (21)

Proof: by Taylor's theorem

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \tag{22}$$

We let x = 1 to (22)

We have

$$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{23}$$

So that

$$L(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \tag{24}$$

Form (7), (24) then (20) become

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \ \ln \tan x dx = \frac{\pi}{4}\psi(1) + L'(1)$$

This concludes the proof of equation (3)

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利用 PS1 函數及狄西里 L 函數探討積分 $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \ln \tan xdx$

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摘 要

利用 PS1 函數及狄里西 L-函數表示

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \ln \tan xdx = \frac{\pi}{4}\psi(1) + L'(1)$$

關鍵字：1.奧衣勒常數
2.加瑪函數
3.狄里西 L-函數
4.PS1 函數
5.Weierstrass 定義的加瑪函數

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