Nonlinear Filter Design for Markovian Jump Nonlinear Stochastic Systems

Wen-Hao Chen, Juh-Shinun Lin

Abstract

This paper discusses the nonlinear filter design for state estimation of Markovian jump nonlinear stochastic systems. Our results extend the filter theory for Markovian jump linear systems to the nonlinear stochastic systems with Markovian jumping parameters. By using the Lyapunov approach to design the nonlinear filter for Markovian jump nonlinear stochastic systems, the state estimation error for the proposed filter can approach to zero asymptotically in probability. Finally, a simulated example is given to justify the performance of the proposed nonlinear filter.

Keywords: Markovian jump, Nonlinear filter, Nonlinear systems, Stochastic system.

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馬爾可夫跳躍型非線性隨機系統的非線性濾波器之設計

陳文豪、林助訓

摘 要

本文是在探討用來作爲馬爾可夫跳躍型非線性隨機系統的狀態估測之非線性濾波器的設計。本文的方法是將馬爾可夫跳躍型線性系統的濾波器理論推廣至具有馬爾可夫跳躍型參數的非線性隨機系統。本文是依據李亞普諾夫方法來爲馬爾可夫跳躍型非線性隨機系統設計非線性濾波器，因而本文所提議的濾波器的狀態估測誤差就機率而言會漸近趨於零。最後，也提供一個使用電腦模擬的例子來加以驗證本文所提議的濾波器之效能。

關鍵詞：馬爾可夫跳躍、非線性濾波器、非線性系統、隨機系統。

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I. INTRODUCTION

This paper discusses the nonlinear filter design for state estimation of Markovian jump nonlinear stochastic systems. The Markovian jump systems are important in practical applications since this kind of systems can be used to represent many important physical systems subject to random failures and structure changes. The filtering for a system is an important topic in the fields of signal processing and control engineering. The filter design for Markovian jump linear system has been discussed [8, 12, 14, 15]. But there were few papers discussing the filter design of Markovian jump nonlinear system. In this paper, we investigate the nonlinear filter design problem for the Markovian jump nonlinear stochastic systems. By using the Lyapunov approach to design the nonlinear filter, the state estimation error for the proposed filter approaches to zero asymptotically in probability. Finally, we demonstrate the usefulness and applicability of the proposed nonlinear filter by means of a numerical simulation example.

For convenience, we adopt the following notations:

A': the transpose of matrix A.

$R_+ := [0,\infty)$: the set of all nonnegative real numbers.

$W(t)$: a one-dimensional standard Wiener process.

$\theta(t)$: the system parameter jumping regime at time t, which is a Markovian state.

$\mathcal{F}_t := \sigma(\{W(s), 0 \le s \le t\})$: the smallest $\sigma-$algebra with respect to which the function $W(s)$, $0 \le s \le t$, is measurable.

$\Theta_t := \sigma(\{\theta(s), 0 \le s \le t\})$:the smallest $\sigma-$algebra with respect to which the function $\theta(s)$, $0 \le s \le t$ is measurable. Throughout this paper we assume that $\sigma-$algebra Θ_t is independent of $\sigma-$algebra $\mathcal{F}_t$ for all $t \ge 0$.

$\mathcal{H}_t = \mathcal{F}_t \vee \Theta_t := \sigma(\mathcal{F}_t \cup \Theta_t)$, i.e., the smallest $\sigma-$algebra with respect to which both functions $W(s)$ and $\theta(s)$, $0 \le s \le t$, are measurable.

$C_2^0(U_0, \Pi)$: the class of functions $V_{\theta(t)}(x)$ which are twice continuously differentiable with respect to $x \in U_0 \subset R^n$ except possibly at the origin, where U_0 is the neighborhood of the origin.

II. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider the following Markovian jump nonlinear stochastic system

$$\begin{cases} dx(t) = f_{\theta(t)}(x(t))dt + h_{\theta(t)}(x(t))dW, \\ Y(t) = s_{\theta(t)}(x(t)) + k_{\theta(t)}(x(t))w \end{cases} \tag{2.1}$$

Hence, $x(t) \in R^n$ is called the system state which is an n-dimensional random vector process defined on a complete probability space $(\Omega, \mathcal{H}, \mathcal{P})$, $Y(t) \in R^{n_Y}$ is the measurement output, $W(t)$ is a one-dimensional standard Wiener process, and w is a one-dimensional white noise with $w = dW/dt$. $f_{\theta(t)}, h_{\theta(t)}, s_{\theta(t)}, k_{\theta(t)}$ are uniformly continuous nonlinear functions with $f_{\theta(t)}(0) = 0$, $h_{\theta(t)}(0) = 0$, $s_{\theta(t)}(0) = 0$, $k_{\theta(t)}(0) = 0$. The system parameter jumping regime $\theta(t)$ is a continuous-time Markov process with finite discrete state-space $\Pi :=$ $\{1, 2, \cdots, N\}$, i.e. $\theta(t) \in \Pi$, and transition rate matrix $\Lambda = [\lambda_{ij}]_{i,j\in\Pi}$, where λ_{ij} are real numbers denoting the transition rate from the system parameter regime i to the system parameter regime j such that $\lambda_{ij} \geq 0$ for $i \neq j$, and $\lambda_{ii} = -\sum_{j\neq i}\lambda_{ij}$ for all $i \in \Pi$. Let us introduce the regime indicator $\phi_t \in R^N$ with components $\phi_{ii} = 1$ if $\theta(t) = i$ and $\phi_{ii} = 0$ otherwise, for $i = 1, 2, \cdots, N$, whose dynamics is given by a Markovian chain [3]

$$d\phi_t = \Lambda'\phi_t dt + dM_t \tag{2.2}$$

where M_t is a Θ_t-martingale.

The transition probability matrix $P(t)$ is related with the transition rate matrix Λ by $P(t)=[p_{ij}(t)]=e^{\Lambda t}$, $t>0$. In other words, the transition probabilities are given by [9]:

$$P[\theta(t+\Delta) = j \mid \theta(t) = i] = \begin{cases} \lambda_{ij}\Delta + o(\Delta) & if \quad j \neq i \\ 1 + \lambda_{ii}\Delta + o(\Delta) & if \quad j = i \end{cases} \tag{2.3}$$

where $o(\Delta)$ are the remainder terms such that $\lim_{\Delta\to 0} o(\Delta)/\Delta = 0$.

The system (2.1) can be used to represent many important physical systems subject to random failures and structure changes, such as electric power systems [16], solar thermal receiver systems [13], communications systems [1], aircraft flight systems [10], and manufacturing systems [2, 3]. Therefore, this kind of system is very important in the field of control and filtering.

To obtain a tractable mathematical interpretation of $Y(t)$ in (2.1), we define [5]

$$y(t) = \int_0^t Y(s)ds \tag{2.4}$$

and thereby obtain the stochastic differential equations for (2.1)

$$\begin{cases} dx(t) = f_{\theta(t)}(x(t))dt + h_{\theta(t)}(x(t))dW, \\ dy(t) = s_{\theta(t)}(x(t))dt + k_{\theta(t)}(x(t))dW \end{cases} \tag{2.5}$$

Remark 2.1:

Note that if $Y(s)$ is known for $0 \le s \le t$, then $y(s)$ is known for $0 \le s \le t$, and vice versa. So no information is lost or gained by considering $y(t)$ as our observations instead of $Y(t)$. But this allows us to obtain a well-defined mathematical model in this situation [11].

Lemma 2.1 [6, 17]:

For a set of positive definite functions $V_i(x(t)) \in C_2^0(R^n, \Pi)$, the infinitesimal generator $\mathcal{L}$ of the system (2.1) or (2.5) is given by

$$\begin{aligned} &\mathcal{L}V_i(x(t)) := \\ &\lim_{\Delta\to 0}\frac{1}{\Delta}[E\{V_{\theta(t+\Delta)}(x(t+\Delta)) \mid x(t), \theta(t) = i\} \\ &-V_{\theta(t)=i}(x(t))] \\ &= \left(\frac{\partial V_i(x(t))}{\partial x}\right)' f_{\theta(t)}(x(t)) \\ &+ \frac{1}{2}h'_{\theta(t)}(x(t))\frac{\partial^2 V_i(x(t))}{\partial x^2}h_{\theta(t)}(x(t)) \\ &+ \sum_{j\in\Pi}\lambda_{ij}V_j(x(t)) \end{aligned} \tag{2.6}$$

Lemma 2.2 [7, 4]:

If there exist a set of positive definite Lyapunov functions $V_i(x(t)) \in C_2^0(U_0, \Pi)$, which solve the following inequalities

$$\begin{cases} \mathcal{L}V_i(x(t)) = \\ = \left(\frac{\partial V_i(x(t))}{\partial x}\right)' f_i(x(t)) \\ \quad + \frac{1}{2}h'_i(x(t))\frac{\partial^2 V_i(x(t))}{\partial x^2}h_i(x(t)) \\ \quad + \sum_{j\in\Pi}\lambda_{ij}V_j(x(t)) < 0 \\ \qquad \forall i \in \Pi,\ \forall x \in U_0,\ x \ne 0,\ \forall t \ge 0 \\ V_i(0) = 0 \qquad \forall i \in \Pi \end{cases} \tag{2.7}$$

then the equilibrium point $x \equiv 0$ of the dynamical system (2.1) or (2.5) is asymptotically stable in probability, i.e. the state $x(t) \to 0$ in probability as $t \to \infty$.

III. NONLINEAR FILTER OF MARKOVIAN JUMP NONLINEAR STOCHASTIC SYSTEMS

In this study, for the nonlinear Markovian jump system (2.1) or (2.5), a filter is proposed in the following for state estimation:

$$\begin{cases} d\hat{x}(t) = f_{\theta(t)}(\hat{x}(t))dt + L_{\theta(t)}(\hat{x}(t))[dy - d\hat{y}] \\ = \{f_{\theta(t)}(\hat{x}(t)) \\ \quad + L_{\theta(t)}(\hat{x}(t))[s_{\theta(t)}(x(t)) - s_{\theta(t)}(\hat{x}(t))]\}dt \\ \quad + L_{\theta(t)}(\hat{x}(t))k_{\theta(t)}(x(t))dW \\ \hat{x}(0) = 0 \end{cases} \tag{3.1}$$

where $L_{\theta(t)}(\hat{x}(t))$ is the filter gain.

Set the augmented state vector $\eta := [x' \ \hat{x}']' \in R^{2n}$, then the augmented state dynamical system for (2.5) and (3.1) is given as follows

$$d\eta(t) = f_{\theta(t),e}(\eta(t))dt + h_{\theta(t),e}(\eta(t))dW \tag{3.2}$$

where

$$\begin{cases} f_{\theta(t),e}(\eta(t)) = \begin{bmatrix} f_{\theta(t)}(x(t)) \\ f_{\theta(t)}(\hat{x}(t)) + L_{\theta(t)}(\hat{x}(t)) \times \\ [s_{\theta(t)}(x(t)) - s_{\theta(t)}(\hat{x}(t))] \end{bmatrix} \\ h_{\theta(t),e}(\eta(t)) = \begin{bmatrix} h_{\theta(t)}(x(t)) \\ L_{\theta(t)}(\hat{x}(t))k_{\theta(t)}(x(t)) \end{bmatrix} \end{cases} \tag{3.3}$$

In the following, we denote $\widetilde{U}_0$ as the neighborhood of the origin $\eta = 0$ in R^{2n}, and e as the state estimation error, i.e., $e = x - \hat{x}$.

Lemma 3.1 [6, 17]:

For a set of positive definite functions $V_i(\eta(t)) \in C_2^0(\widetilde{U}_0, \Pi)$, the infinitesimal generator $\mathcal{L}$ of the system (3.2) is given by

$$\begin{aligned} &\mathcal{L}V_i(\eta(t)) := \\ &\lim_{\Delta \to 0} \frac{1}{\Delta}[E\{V_{\theta(t+\Delta)}(\eta(t+\Delta)) \mid \eta(t), \theta(t) = i\} \\ &- V_{\theta(t)=i}(\eta(t))] \\ &= \left(\frac{\partial V_i(\eta(t))}{\partial \eta}\right)' f_{\theta(t),e}(\eta(t)) \\ &\quad + \frac{1}{2} h'_{\theta(t),e}(\eta(t)) \frac{\partial^2 V_i(\eta(t))}{\partial \eta^2} h_{\theta(t),e}(\eta(t)) \\ &\quad + \sum_{j \in \Pi} \lambda_{ij} V_j(\eta(t)) \end{aligned} \tag{3.4}$$

Proof: The result follows immediately from Lemma 2.1. □

Theorem 3.1:

If there exist a set of positive definite Lyapunov functions $V_i(\eta(t)) \in C_2^0(\widetilde{U}_0, \Pi)$ and filter gains $L_i(\hat{x}(t))$, which solve the following inequalities

$$\begin{cases} \left(\dfrac{\partial V_i(\eta(t))}{\partial \eta}\right)' \begin{bmatrix} f_i(x(t)) \\ f_i(\hat{x}(t)) + L_i(\hat{x}(t)) \times \\ [s_i(x(t)) - s_i(\hat{x}(t))] \end{bmatrix} \\ + \dfrac{1}{2} \begin{bmatrix} h_i(x(t)) \\ L_i(\hat{x}(t)) k_i(x(t)) \end{bmatrix}' \times \dfrac{\partial^2 V_i(\eta(t))}{\partial \eta^2} \times \\ \begin{bmatrix} h_i(x(t)) \\ L_i(\hat{x}(t)) k_i(x(t)) \end{bmatrix} + \sum_{j \in \Pi} \lambda_{ij} V_j(\eta(t)) < 0 \\ \qquad \forall i \in \Pi, \forall \eta \in \widetilde{U}_0, \eta \neq 0, \forall t \geq 0 \\ V_i(0) = 0, \quad \forall i \in \Pi \end{cases} \tag{3.5}$$

then the state estimation error $e(t) \to 0$ in probability as $t \to \infty$.

Proof: From (3.3) and (3.4), we have

$$\begin{aligned} &\mathcal{L}V_i(\eta(t)) \\ &= \left(\frac{\partial V_i(\eta(t))}{\partial \eta}\right)' f_{i,e}(\eta(t)) \\ &\quad + \frac{1}{2} h'_{i,e}(\eta(t)) \frac{\partial^2 V_i(\eta(t))}{\partial \eta^2} h_{i,e}(\eta(t)) \\ &\quad + \sum_{j \in \Pi} \lambda_{ij} V_j(\eta(t)) \\ &= \left(\frac{\partial V_i(\eta(t))}{\partial \eta}\right)' \begin{bmatrix} f_i(x(t)) \\ f_i(\hat{x}(t)) + L_i(\hat{x}(t)) \times \\ [s_i(x(t)) - s_i(\hat{x}(t))] \end{bmatrix} \\ &+ \frac{1}{2} \begin{bmatrix} h_i(x(t)) \\ L_i(\hat{x}(t)) k_i(x(t)) \end{bmatrix}' \times \frac{\partial^2 V_i(\eta(t))}{\partial \eta^2} \times \\ &\begin{bmatrix} h_i(x(t)) \\ L_i(\hat{x}(t)) k_i(x(t)) \end{bmatrix} + \sum_{j \in \Pi} \lambda_{ij} V_j(\eta(t)) \\ &\qquad \forall i \in \Pi \end{aligned} \tag{3.6}$$

Then, from (3.5) and (3.6), it follows that

$$\begin{cases} \mathcal{L}V_i(\eta(t)) < 0 \\ \forall i \in \Pi, \forall \eta \in \widetilde{U}_0, \eta \neq 0, \forall t \geq 0 \end{cases} \tag{3.7}$$

Then, by Lemma 2.2, it follows from (3.7) that the equilibrium point $\eta \equiv 0$ of the dynamical system (3.2) is asymptotically stable in probability, i.e., $\eta := [x' \; \hat{x}']' \to 0$ in probability as $t \to \infty$. So it is obviously true that the state estimation error $e(t) \to 0$ in probability as $t \to \infty$.

This completes the proof. □

Remark 3.1:

There are many kinds of functions which can be chosen to be $V_i(\eta(t))$ and $L_i(\hat{x}(t))$ in (3.5). But the most important thing is that functions $V_i(\eta(t))$ and $L_i(\hat{x}(t))$ should be suitably chosen to make the inequalities in (3.5) solvable. So we first choose some kinds of appropriate functions $V_i(\eta(t))$ and $L_i(\hat{x}(t))$ with their coefficients to be determined. Next, after substituting into (3.5) the values of η for many points which are densely and uniformly distributed within the region $\widetilde{U}_0$, the inequalities in (3.5) become a set of inequalities with the to-be-determined coefficients of appropriate functions $V_i(\eta(t))$ and $L_i(\hat{x}(t))$. Then the set of inequalities above can be solved to find filter gains $L_i(\hat{x}(t))$ with Matlab Optimization Toolbox. □

IV. AN ILLUSTRATIVE EXAMPLE AND SIMULATIONS

In the previous section, the nonlinear filter design problem has been discussed. In this section, a simple example is given to illustrate the design procedure and to justify the performance of the proposed nonlinear filter with Matlab simulation.

Consider the following Markovian jump nonlinear stochastic system

$$\begin{cases} \begin{cases} dx = -x^3 dt + x^2 dW \\ dy = 10x dt + 2x^2 dW \end{cases} & for\ \theta(t) = 1 \\ \\ \begin{cases} dx = -x^5 dt + x^2 dW \\ dy = 10x dt + x^3 dW \end{cases} & for\ \theta(t) = 2 \end{cases} \tag{4.1}$$

with the transition rate matrix $\Lambda = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$.

The goal is to design a nonlinear filter for the system (4.1).

Now we follow the procedures in Remark 3.1 to solve the inequalities in (3.5). The positive definite functions $V_i(\eta)$ are specified as

$V_1(\eta) = \eta' P_1 \eta$,

$V_2(\eta) = \eta' P_2 \eta$,

the filter gains are specified as

$L_1(\hat{x}) = L_{1a}\hat{x}^2 + L_{1b}\hat{x} + L_{1c}$,

$L_2(\hat{x}) = L_{2a}\hat{x}^2 + L_{2b}\hat{x} + L_{2c}$,

the initial state and the initial estimated state are specified as

$x(0) = 5$, $\hat{x}(0) = 0$,

and $\widetilde{U}_0 = \{\eta \,|\, -1 \le x \le 10, -1 \le \hat{x} \le 10\}$.

After solving the inequalities in (3.5) with Matlab Optimization Toolbox, we get $P_1 = \begin{bmatrix} 0.25116 & 0 \\ 0 & 0.00001 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0.2523 & 0 \\ 0 & 0.00001 \end{bmatrix}$, and the filter gains $L_{1a} = 0.1$, $L_{1b} = 0.1$, $L_{1c} = 30.5$, $L_{2a} = 0.1$, $L_{1b} = 5.473$ and $L_{1c} = 47.365$.

The simulation results of the system in (4.1) with the proposed nonlinear filter are given in Fig. 1. The system parameter jumping regime $\theta(t)$ is shown in Fig. 1(a) and Fig. 1(b) shows $x(t)$ and $\hat{x}(t)$.

Remark 4.1:

As is well known, the filter gain in the linear system is always chosen to be a constant, i.e. L. To emphasize the filter in this paper is a nonlinear filter, the filter gains of nonlinear form, i.e. $L_1(\hat{x}) = L_{1a}\hat{x}^2 + L_{1b}\hat{x} + L_{1c}$ and $L_2(\hat{x}) = L_{2a}\hat{x}^2 + L_{2b}\hat{x} + L_{2c}$ are chosen in the above example. Of course, as mentioned in Remark 3.1, there are also many other kinds of functions $V_i(\eta(t))$ and $L_i(\hat{x}(t))$ which can be chosen to make the inequalities in (3.5) solvable.

V. CONCLUSION

In this paper, we investigated the nonlinear filter design problem for the Markovian jump nonlinear stochastic systems. By using the Lyapunov approach to design the nonlinear filter, the state estimation error for the proposed filter approaches to zero asymptotically in probability. Finally, a simulated example was given to justify the performance of the proposed nonlinear filter.

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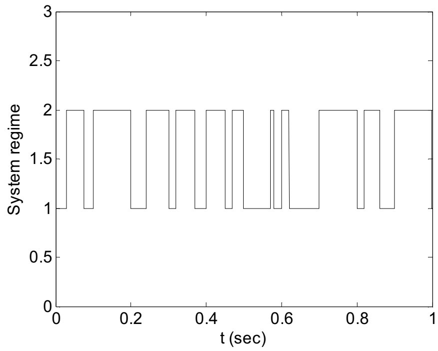
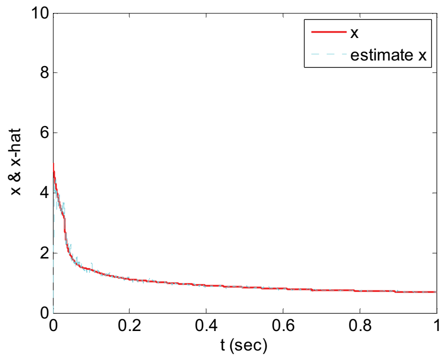


(a) System parameter jumping regime $\theta(t)$.



(b) State $x(t)$ and its estimate $\hat{x}(t)$.

Fig. 1. Computer simulation in the numerical example.