# Linear stability characterization of thin Newtonian liquid films flowing down a cylinder moving in a vertical direction

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## Abstract

This project presents a stability analysis of thin Newtonian liquid films flowing down a cylinder moving in a vertical direction. The long-wave perturbation method is employed to derive the generalized kinematic equations for a free film interface. The current thin liquid film stability analysis provides a valuable input to investigations into the influence of the style of motion of the vertical cylinder on the stability behavior of the thin film flow.

**Keywords:** Newtonian liquid, thin film flow, long-wave perturbation.

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# 沿垂直方向移動之直立圓柱表面流下的牛頓流體薄膜流的線性穩定性分析

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## 摘 要

本文針對牛頓流體薄膜流，探討沿垂直方向移動的直立圓柱表面流下的薄膜流之線性液動穩定性問題。首先使用長波微擾法推導薄膜的自由面方程式，在探討薄膜流場的穩定性問題上，主要分析圓柱的移動效應對系統穩定性的影響。

**關鍵詞**：牛頓流體、薄膜流、長波微擾法。

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# 1. Introduction

The stability characterization of film flows traveling down along a vertical cylinder is of great importance to the quality control of many industrial products. Thus, the research effort made toward improvement on this matter has been emerged as a subject of great interest to numerous worldwide researchers in past decades. Typical application examples can be found across different industrial sectors including mechanical, chemical and nuclear engineering. It is well known that the stability controls are generally required in precision finishing processes of coating, laser cutting, and casting. Since macroscopic instability can cause disastrous conditions to film flows and thus very detrimental to the needed quality of final products, it is highly desirable to develop suitable working conditions for homogeneous film growth to adapt to various flow configurations and associated time-dependent properties.

Detailed reviews on linear stability theories for various film flows has formally presented by Lin [1] and Chandrasekhar [2]. The Landau equation was re-derived in 1956 by Stuart [3] using the disturbed energy balance equation and Reynolds stresses. Benjamin [4] and Yih [5] formulated the perturbed wave equation for free surface flows. The stability behaviors of flows having long disturbed wave were fully studied in this paper and some significant observations on film flows over an inclined plane are obtained. These observations include (1) the flow that is disturbed by a longer wave is less stable than that of the flow disturbed by a shorter wave; (2) the film flow becomes less stable as the inclined angle increases; (3) the film flow traveling down along a vertical plate becomes unstable as the critical Reynolds number becomes nearly zero; (4) the film flow becomes somehow stabilized as the surface tension of the film increases; (5) velocity of the unstable long disturbed wave is approximately twice of the wave velocity on the free surface. The effect of surface tension was found by many researchers [6-8] as one of the necessary conditions that lead to the solution of supercritical stability in analyzing this type of problems. The effect of surface tension on flow stability was considered significant by Lin [6], Nakaya [7], and Krishna et al. [8]. Renardy et al. [9] and Tsai et al. [10] presented the work of both linear and nonlinear stability analysis for a film flow traveling down along an inclined or a vertical plate. Detailed flow analysis was found of great importance in the development of stability theory for characterizing the behaviors of various film flows.

After careful literature review on the papers, it was found that the stability of thin Newtonian film flows moving along vertical cylinder appeared to be very important in various coating, painting, surface drawing and lubrication processes. This type of stability problems has not yet been fully explored so far in the literature. The types of stability problems are indeed of great importance for many industrial applications. In this paper, the finite-amplitude stability of a thin Newtonian film flow traveling down along a vertical quiescent, up-moving, and down-moving cylinder is thoroughly investigated. The influence of the cylinder moving styles on the equilibrium finite amplitude is studied and characterized. Several numerical examples are presented to verify the computational results and also to illustrate the effectiveness of the proposed modeling approach.

## 2. Generalized Kinematic Equation

Fig.1 shows the configuration of a thin Newtonian film flow traveling down along a vertically moving cylinder. The governing equations can be expressed in terms of cylindrical coordinates $(r^*, \ z^*)$ as

$$\frac{1}{r^*}\frac{\partial(r^* u^*)}{\partial r^*}+\frac{\partial w^*}{\partial z^*}=0 \tag{1}$$

$$\rho(\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial r^*}+w^*\frac{\partial u^*}{\partial z^*}) =\frac{1}{r^*}\frac{\partial(r^*\tau_{r^*r^*})}{\partial r^*}+\frac{\partial \tau_{z^*r^*}}{\partial z^*}-\frac{1}{r^*}\tau_{\theta^*\theta^*} \tag{2}$$

$$\rho(\frac{\partial w^*}{\partial t^*}+u^*\frac{\partial w^*}{\partial r^*}+w^*\frac{\partial w^*}{\partial z^*}) =\frac{1}{r^*}\frac{\partial(r^*\tau_{r^*z^*})}{\partial r^*}+\frac{\partial \tau_{z^*z^*}}{\partial z^*}+\rho g \tag{3}$$

Individual stress components can be expressed in terms of velocity gradient and flow pressure as

$$\tau_{r^*r^*}=-p^*+2\mu_0\frac{\partial u^*}{\partial r^*} \tag{4}$$

$$\tau_{z^*z^*}=-p^*+2\mu_0\frac{\partial w^*}{\partial z^*} \tag{5}$$

$$\tau_{r^*z^*}=\tau_{z^*r^*}=\mu_0(\frac{\partial u^*}{\partial z^*}+\frac{\partial w^*}{\partial r^*}) \tag{6}$$

$$\tau_{\theta^*\theta^*}=-p^*+2\mu_0\frac{u^*}{r^*} \tag{7}$$

where $u^*$ and $w^*$ are velocity components in $r^*$ and $z^*$ directions, respectively. p is the flow pressure, $\rho$ is the film density, and $\mu_0$ is the dynamic viscosity. The boundary conditions for the film flow system at the cylinder surface of $r^*=R^*$ can be expressed as

$$u^*=0 \tag{8}$$

$$w^*=V_w^* \tag{9}$$

where $V_w^*$ is the moving velocity of the vertical cylinder. The boundary conditions for the film flow at free surface of $r^* = R^* + h^*$ are derived based on the results given by Edwards et al. [11]. The shear stress for film flow at free surface is given as

$$\frac{\partial h^*}{\partial z^*}[1+(\frac{\partial h^*}{\partial z^*})^2]^{-1}(\tau_{r^*r^*}-\tau_{z^*z^*}) + [1-(\frac{\partial h^*}{\partial z^*})^2][1+(\frac{\partial h^*}{\partial z^*})^2]^{-1}\tau_{r^*z^*}=0. \quad (10)$$

The normal stress for film flow at free surface is given as

$$[1+(\frac{\partial h^*}{\partial z^*})^2]^{-1}[2\tau_{r^*z^*}\frac{\partial h^*}{\partial z^*}-\tau_{r^*r^*} -\tau_{z^*z^*}(\frac{\partial h^*}{\partial z^*})^2]+S^*\{\frac{\partial^2 h^*}{\partial z^{*2}}[1 +(\frac{\partial h^*}{\partial z^*})^2]^{-3/2}-\frac{1}{r^*}[1+(\frac{\partial h^*}{\partial z^*})^2]^{-1/2} = Pa^*. \quad (11)$$

The kinematic condition that the flow velocity normal to a free surface is naught can be given as

$$\frac{\partial h^*}{\partial t^*}+\frac{\partial h^*}{\partial z^*}w^*-u^*=0 \quad (12)$$

where $p_a^*$ is the ambient pressure, $S^*$ is the surface tension, $h^*$ is the local film thickness. The variable associated with a superscript " * " stands for a dimensional quantity. By introducing the stream function $\varphi^*$, the dimensional velocity components can now be expressed as

$$u^*=\frac{1}{r^*}\frac{\partial\varphi^*}{\partial z^*},\quad w^*=-\frac{1}{r^*}\frac{\partial\varphi^*}{\partial r^*} \quad (13)$$

In order to minimize the flow variables and to simplify the analysis procedure, it is customary to define dimensionless variables as

$$z=\frac{\alpha z^*}{h_0^*},\quad r=\frac{r^*}{h_0^*},\quad t=\frac{\alpha u_0^* t^*}{h_0^*},$$

$$h=\frac{h^*}{h_0^*},\quad \varphi=\frac{\varphi^*}{u_0^* h_0^{*2}},\quad p=\frac{p^*-p_a^*}{\rho u_0^{*2}}$$

$$\mathrm{Re}=\frac{u_0^* h_0^*}{\nu_0},\quad S=(\frac{S^{*3}}{2^4\rho^3\nu_0{}^4 g})^{1/3},$$

$$\alpha=\frac{2\pi h_0^*}{\lambda},\quad R=\frac{R^*}{h_0^*}, \quad (14)$$

The moving velocity of the vertical cylinder can then be expressed as

$$V_w^* = N u_0^* \quad (15)$$

where N is a specific constant ratio of the cylinder velocity to the free stream velocity. The reference velocity can then be expressed as

$$u_0^*=(1+N)\frac{g h_0^{*2}}{4\nu_0\Gamma} \quad (16)$$

and

$$\Gamma=[2(1+R)^2\ln(\frac{1+R}{R})-(1+2R)]^{-1} \quad (17)$$

Since the modes of long-wavelength that gives the smallest wave number are most likely to induce flow instability for the film flow [4,5], the dimensionless wave number of the long-wavelength mode, $\alpha$, is then chosen as the perturbation parameter for variable expansion. By so doing the stream function and flow pressure can be perturbed and represented as