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Finite Deformation of 2-D Thin Circular Curved Beams

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Abstract

An analytical method is derived for obtaining the finite deformation of 2-D thin curved beams with circular curvatures. The general solutions are expressed by fundamental geometric quantities. As the radius of curvature is given, the fundamental geometric quantities can be calculated to obtain the closed form solutions of the axial force, shear force, bending moment, rotation angle, and deformed and un-deformed displacement fields. The closed-form solutions of the circular beams under pure bending moment cases and simple circular curved beam under a pair of horizontal forces are presented. The results show the consistency in comparison with ANSYS results.

Keywords: Finite deformation, Curved beams, Analytical solutions, Nonlinear behavior.

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二維圓形薄曲樑之有限變形研究

林秋文

摘 要

本研究應用解析方法，分析研究二維圓形薄曲樑之有限變形；其一般通解以曲樑之基本幾何特性值表示之。當曲率半徑確定時，則曲樑之基本幾何特性值可以計算出；並藉以求得曲樑之剪力、軸向力、彎矩、旋轉角、變形位移場與未變形位移場等物理量之閉合型式解。本文發表了懸臂圓形薄曲樑承受純彎矩作用與簡支承圓形薄曲樑承受水平集中作用力作用下之閉合型式解，並與有限元素法套裝分析軟體 ANSYS 分析結果比較；結果非常一致。

關鍵詞：有限變形理論、曲樑、解析解、非線性行爲。

林秋文：修平技術學院通識教育中心講師

1. Introduction

The rod theory is one of the most developed parts of the elasticity theory. The finite deformation of rods in space, are always related nonlinear geometric behavior. There are two approaches which are very common. One is three dimensional rod theory. The rod is treated as a three dimensional elastic body to which the methods of three dimensional elasticity theory are applied (Green and Naghdi [1]). The other is one dimensional director theory. The rod is treated as a curve (Green and Naghdi [2]). Naghdi [3], Green [4] showed the nonlinear behavior of rods in both ways. Green [5] showed some relationship between two approaches.

Due to the complexity of mathematical models, most studies have to adopt some kind of simplification, such as small displacement, small shearing strain, small rotation or small shearing effect. By using finite element method, Li. [6,7] derived a finite deformation theory based on total Lagrangian description for 2-D and 3-D beams of zero Poisson's ratio without all the simplifications. Some studied the finite deformation under dynamic loading. Oguibe [8] investigated the numerical study of the elastic plastic response of multilayer aluminum cantilever beams subjected to an impulse loading. The numerical results were compared with the experimental results. Attard [9] studied finite strain of an isotropic hyper-elastic Hookean beam. He used an appropriate strain energy density. The shear effect was included. The solution was also applied to stability behavior design of a helical spring. Mauget [10] applied Lagrangian coordinates to derive an isotropic constitutive law for a large displacement formulation of woods. Toi [11] used total Lagrangian approach for the super-elastic large deformation analysis of a shape memory alloy helical springs.

Most studies focus on straight rods. Only few investigate curve rods. Atanackovic [12] analyzed the finite deformation of a circular ring under uniform pressure. Brush [13] derived a finite deformation stability equation for circular ring under various pressures. He also investigated the stability of nonlinear equilibrium equations for fluid pressure loading. Due to the complexity of mathematical models, analytical solutions are very limited. Timoshenko [14] showed the large deformation of an elastica. It also showed the stability of a straight beam of large deformation. In this paper, we apply the tangent slope coordinate theory by Lin [15, 16, 17] to study finite deformation 2-D of curved rods with variable curvatures.

2. Fundamental Equations

Consider an elastic curved beam of variable curvature whose axis lies on a 2-D plane. Assume the rod is made of elastic material such that the stress is linear to the strain even for finite deformation of the beam. Since the strain is finite, the displacement of a point, the extension of the axis and the rotation angle of any cross section are not necessarily small. To simplify the analysis, assume cross sections do not change the shape and size and the cross section is always orthogonal to the axis in the deformed state.

To describe the curve beam on a 2-D reference configuration, the un-deformed length element dS after deformation becomes the deformed length element ds.The coordinate of end point (X,Y) in the un-deformed state deforms to (x,y) shown as Fig. 1. At the un-deformed state, the tangent slope angle at (X,Y) is denoted by α. At the deformed state, the tangent slope at (x,y) is denoted by θ. The deformation at (X,Y) is denoted by (u,v) where u is the horizontal displacement, and v is the vertical displacement. Hence

$$x = X + u, \quad y = Y + v. \tag{1}$$

The rotation angle φ can be found by

$$\varphi = \theta - \alpha. \tag{2}$$

Since the strain at the centroid axis is defined by $\varepsilon=(ds-dS)/dS$, or

$$ds = (1+\varepsilon)dS. \tag{3}$$

As in the case of in-extensional curved beam, $\varepsilon=0$. For any length element dS, there is a corresponding radius of curvature R, such that

$$dS = Rd\alpha. \tag{4}$$

Here the radius of curvature R does not have to be a constant. Most well known curves can be determined by specifying the radius of curvature, such as circle, ellipse, parabola, cycloid, hyperbola, catenary, spiral curves, etc.

For the deformed length element ds, the corresponding radius of curvature is denoted by r, i.e.

$$ds = rd\theta. \tag{5}$$

At a distance z from centroid axis, the un-deformed length element is denoted by

$$dS_z = (R+z)d\alpha \tag{6}$$

and the deformed length element is

$$ds_z = (r+z)d\theta \tag{7}$$

The strain at a distance of z defined by

$$\varepsilon_z = \frac{ds_z - dS_z}{dS_z}. \tag{8}$$

Equation (8) can be simplified to

$$\varepsilon_z = \varepsilon + z\frac{d\varphi}{dS}. \tag{9}$$

Here assume $z<<R$ so that $z/R<<1$ can be neglected. In other words, the curved beam is slender in the sense that dimension of cross section is much less than the dimension of radius of curvature.

The axial force N is defined as

$$N=\int_A E\varepsilon_z dA = EA\varepsilon \qquad (10)$$

where A denotes the cross sectional area at any length. Here assume A to be a constant. The moment M is defined as

$$M=\int_A E\varepsilon_z z dA = EI\frac{d\varphi}{dS} \qquad (11)$$

where I is the moment of inertia of the cross section with respect to centroid axis.

Assume I is a constant. The notation and sign convention of axial force N, moment M together with shear force Q, external distributed tangential force q_α and radial force q_R are shown in Fig. 2. The force balance in the reference configuration can be expressed by

$$\frac{dN}{dS}+\frac{Q}{R}=-q_\alpha,\ -\frac{N}{R}+\frac{dQ}{dS}=-q_R,\ \frac{dM}{dS}=Q. \qquad (12)$$

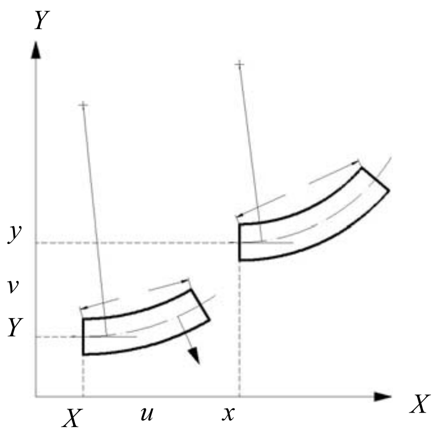

Fig. 1. Deformation of length element

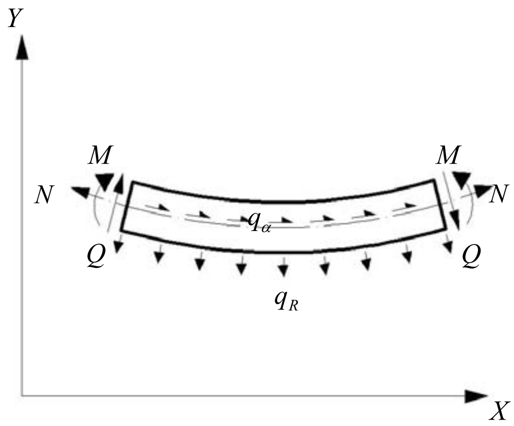

Fig. 2. Sign convention of forces and moment

The three equations show the balance of forces along tangential direction, radial direction and moment. The equilibrium equations can be obtained by taking free body of a curved element. These equations are the same as the force balance equation in small deformation [15], since the effect of finite deformation does not effect equilibrium equation in the reference configuration.

The equilibrium equations can also be expressed by deformed configuration,

$$\frac{dN}{ds}+\frac{Q}{r}=-\frac{q_\alpha}{1+\varepsilon},\quad -\frac{N}{r}+\frac{dQ}{ds}=-\frac{q_R}{1+\varepsilon},$$
$$\frac{dM}{ds}=\frac{Q}{1+\varepsilon}. \tag{13}$$

Here assume the external distributed loads q_α, q_R change direction but not in magnitude. There are several ways to describe equilibrium equations. For instance all forces can be expressed along X, Y directions. Equations (12) are exactly the same as the equation of equilibrium equations in [14]. So far there are five equations : three equations of equilibrium Eqs. (12) and two constitutive equations Eqs. (10, 11) for N, Q, M, and ε, φ. To complete the analysis of finite deformation, the displacements are included

$$dx=(1+\varepsilon)\cos\theta dS,\quad dy=(1+\varepsilon)\sin\theta dS. \tag{14}$$

Once taking the integrals of Eqs. (14), from Eqs. (1), the displacements u, v can be found.

To carry out the analysis, all the variable are nondimensionalized. Select a characteristic radius of curvature R_o. All the length variables x, y, X, Y, u, v, s, S, r, R, are nondimensionlized with respect to R_o. The axial force N and shear force Q are nondimensionlized with respect to EI/R_o^2, moment M is nondimensionalized with respect to EI/R_o, distributed loads q_α, q_R are scaled with respect to EI/R_o^3.

Due to the complicated term ε which shows the difference of deformed and un-deformed state, the analytical solutions are very limited. Here use Eqs. (13) to evaluate the curved beam. To derive the solution, in the absence of distributed loads, i.e. $q_\alpha=0$, $q_R=0$, Eqs. (13) can be written as

$$\frac{dN}{ds}+\frac{Q}{r}=0,\quad -\frac{N}{r}+\frac{dQ}{ds}=0,\quad \frac{dM}{dS}=Q. \tag{15}$$

By using Eq. (5) and changing variable from s to θ, Eqs. (15a, b) can be combined and solved as

$$N=A_1\cos\theta+A_2\sin\theta$$
$$Q=A_1\sin\theta-A_2\cos\theta \tag{16}$$

where A_1, A_2 are two constants to be determined by boundary conditions. As in the case of in-extensional curved beam, $\varepsilon=0$. Eq. (15c) with the help of Eq. (16b) after integrating once to obtain

$$M=A_1y-A_2x+A_3 \tag{17}$$

where x, y are calculated by

$$x=\int_0^\theta r\cos\bar{\theta}d\bar{\theta},\quad y=\int_0^\theta r\sin\bar{\theta}d\bar{\theta}. \tag{18}$$

There should be two constants, but the origin can be selected as $\theta=0$, $x=y=0$. In Eq. (17), A_3 is a constant to be determined by suitable boundary conditions.

With the help of Eqs. (2, 11) the deformed angle θ can be expressed as by using reference configuration

$$\frac{d\theta}{dS}=\frac{1}{R}+A_1y-A_2x+A_3. \tag{19}$$

By integrating Eq. (19) once, the solution of θ yields

$$\theta=\alpha+A_3S+A_1\int_0^\alpha\frac{yRd\alpha}{1+Z(A_1\cos\theta+A_2\sin\theta)}-A_2\int_0^\alpha\frac{xR}{1+Z(A_1\cos\theta+A_2\sin\theta)}d\alpha+\theta_0 \tag{20}$$

where the strain with the help of Eq. (10)

$$\varepsilon=Z(A_1\cos\theta+A_2\sin\theta) \tag{21}$$

is used in Eq. (20). In Eqs. (20, 21), the dimensionless constant Z is defined as

$$Z=\frac{I}{AR_0^2} \tag{22}$$

which is the square ratio of radius of moment of inertia of the cross section to reference radius of the curve. It is small as the curved beam is slender. In the case of thick beam, Z is usually not small. This constant sometimes is used to distinguish the slender curved beam from thick curved beams. In this paper, Z is small, otherwise Eqs. (10, 11) are no longer valid. As Z approaches zero, it is equivalent to in-extensional assumption.

Once the radius is given, the arc length can be determined by Eq. (4). The constant θ_0 denotes the rotation at $\alpha=0$. Since the deformed coordinates x, y are needed to be solved, it depends on whether the integral can be integrated explicitly. Here in this paper, the focus will be for the analytical form.

3. Applications

Consider a cantilever curved beam of variable curvature. The curve starts from fixed end $\alpha=0$ to free end $\alpha=\beta$ shown as Fig. 3a. The origin is located at $\alpha=0$.

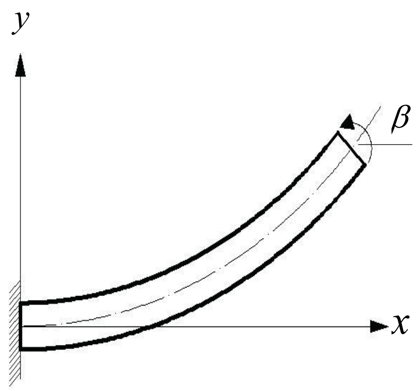

Fig. 3a. A cantilever curved beam under a concentrated moment M_0.

A concentrated nondimensional moment M_0 is applied at the end of $\alpha=\beta$. It is equivalent to a free curved beam which is symmetric with respect to y-axis. The curved beam starts from free end $\alpha=-\beta$, to $\alpha=\beta$. A pair of concentrated moments M_0 is applied at both ends shown as Fig. 3b.

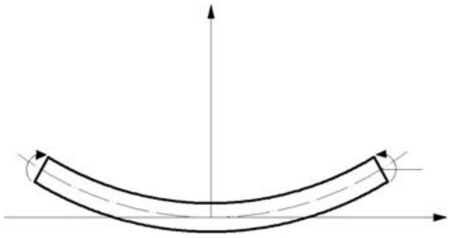

Fig. 3b. Simple curved beams under a pair of moment M_0.

Due to symmetry, only the portion from $\alpha=0$ to $\alpha=\beta$ is needed to be considered. Due to fixed end or symmetry, at $\alpha=0$ the boundary conditions are

$$\varphi=0,\ u=0,\ v=0, \quad (23)$$

$$\theta=0,\ x=0,\ y=0. \quad (24)$$

At the free end $(\alpha=\beta)$, the axial force, shear force and moment are

$$N(\beta)=0,\ Q(\beta)=0,\ M(\beta)=M_0. \quad (25)$$

Substituting Eqs. (25) into Eqs. (16, 17), the constants can be found $A_1=A_2=0$, $A_3=M_0$. Substituting Eq. (25a) into Eq. (20), the deformed angle of slope becomes

$$\theta=\alpha+kS,\quad k=\frac{M_0R_0}{EI}. \quad (26)$$

Integrate Eq. (14) once to yield

$$x=\int_0^\alpha R\cos(\alpha+kS)d\alpha,\quad y=\int_0^\alpha R\sin(\alpha+kS)d\alpha. \quad (27)$$

In Eq. (27), the constants from integrating vanish because at $\alpha=0$, $x=y=0$. Once the radius of curvature of the curve beam is specified, the deformed coordinates can be found.

The displacements u , v with the help of Eqs. (1) are

$$v=\int_0^\alpha 2R\,sin\left(\frac{1}{2}kS\right)cos\left(\alpha+\frac{1}{2}kS\right)d\alpha.$$
$$u=\int_0^\alpha 2R\sin\left(\alpha+\frac{1}{2}kS\right)\sin\frac{kS}{2}d\alpha, \quad (28)$$

Note that the radius of curvature r after deformation can be calculated by taking derivatives of Eqs. (27). It can be simplified to

$$r=\frac{R}{1+kR}. \quad (29)$$

It is seen that if the curve is a circle, or R is a constant, then r is a constant as well only changes magnitude. The magnitude factor is $1/(1+kR)$. However for a curve of variable curvature, R is not a constant. The deformed radius of curvature changes to another function by $1/(1+kR)$. Therefore the curve changes type as it is deformed.

3.1 Circular curve beam under pure bending moments

For a circular beam, choose the radius

as the characteristic radius, i.e.

$$R = 1. \tag{30}$$

It is equivalent to $X=\sin\alpha$, $Y=1-\cos\alpha$ in parametric form. The solution of displacements u, v in Eqs. (28) can be integrated to yield

$$u = \frac{\sin(1+k)\alpha}{1+k} - \sin\alpha, \quad v = -\frac{k}{1+k} + \cos\alpha - \frac{\cos\alpha}{1+k}. \tag{31}$$

The deformed coordinates from Eqs. (27) are

$$x = \frac{1}{1+k}\sin(1+k)\alpha, \quad y = \frac{1}{1+k}\left[1 - \cos(1+k)\alpha\right] \tag{32}$$

The coordinates of Eq. (32) are a circle in parametric form. The radius of curvature is $1/(1+k)$. In other words as the moment M_0 increases, the radius of curvature decreases by the scale of $1/(1+k)$. Since the strain at the centroid axis is zero, the circumference can be calculated to close the circular curve. The required moment to close the circle is

$$M_{o_{close}} = \frac{\pi}{\beta} - 1. \tag{33}$$

3.2 ANSYS results

Consider a cantilever circular curved beam subjected to a pure bending moment $M_0=EI/R_o$, the curve starts from fixed end $\alpha=0$ to free end $\alpha=\pi/2$. Beams have $B\times H$ rectangular cross-section, Material properties are Poisson's ratio and Young's modulus, which $\upsilon=0.3$ and $E=2.2\times10^{11}$. Using ANSYS large deformation static analysis, the various B/H versus deformed displacements u is shown in Table 1 and the various R/B versus deformed displacements u is shown in Table 2.

It shows the consistency of the results of present study with those by ANSYS.

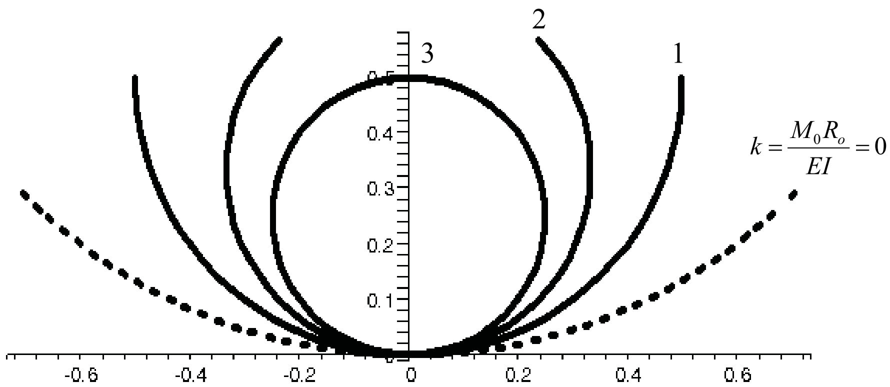

Fig. 4. A circular curved beam from $-\pi/4$ to $\pi/4$ under a couple of moment M_0.

Table 1. The various *B/H* versus deformed displacements *u*. where *R/B* =*1000*.

B/H	*2*	*1*	*1/2*	*1/6*	*1/12*	*1/20*	*1/120*
Error	0.0249%	0.0249%	0.0249%	0.0272%	0.0013%	0.0014%	0.0559%

Table 2. The various *R/B* versus deformed displacements *u*. where *B/H* =*1/12*.

R/B	*2000*	*1250*	*1000*	*800*	*500*	*100*	*10*
u	0.999737	0.999749	0.999987	0.999751	0.999751	0.999751	0.999751
Error	0.0263%	0.0251%	0.0013%	0.0249%	0.0249%	0.0249%	0.0249%

4. Simple curved beam under a pair of horizontal forces

The boundary condition can take at the symmetry point *α=0*,

$$N(0) = -F, \quad Q(0) = 0. \tag{34}$$

Substituting Eqs. (34) into Eqs. (16), the axial and shear forces are

$$N = -F\cos\theta, \quad Q = -F\sin\theta. \tag{35}$$

With the help of Eqs. (15c, 19), the rotation angle can be expressed as

$$\frac{d^2\varphi}{dS^2} = -\lambda^2\sin\theta + \lambda^4 Z\sin\theta\cos\theta \tag{36}$$

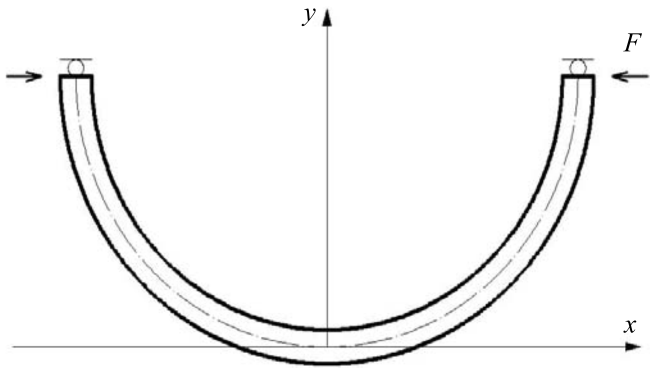

Fig. 5. Simple curved beams under a pair of horizontal forces *F*

where

$$\lambda^2 = \frac{FR_0^2}{EI}. \tag{37}$$

In term of deformed angle, by using Eq. (2), Eq. (36) can be rewritten as

$$\frac{1}{R^2}\frac{d^2\theta}{d\alpha^2} - \frac{1}{R^3}\frac{dR}{d\alpha}\frac{d\theta}{d\alpha} + \frac{1}{R^3}\frac{dR}{d\alpha} = -\lambda^2 \sin\theta + \lambda^4 Z \sin\theta\cos\theta \tag{38}$$

which is a nonlinear differential equation of deformed angle θ, with respect to un-deformed angle α. Once R is assigned in terms of α, two variables θ, α are mixed together. Here an analytical solution will be demonstrated.

Let the curved beam be an in-extensional circular beam of

$$R = 1. \tag{39}$$

Since Z approaches zero for an in-extensional beam, Eq. (38) can be reduced to

$$\frac{d^2\theta}{d\alpha^2} = -\lambda^2 \sin\theta \tag{40}$$

Integrating Eq. (40) once, it yields

$$\frac{1}{2}\left(\frac{d\theta}{d\alpha}\right)^2 = \lambda^2 \cos\theta + C. \tag{41}$$

To find the constant C, the boundary conditions of displacements are needed. Due to symmetry with respect to y-axis, at $\alpha=0$, there is no deformed angle, i.e.

$$\theta(0) = 0. \tag{42}$$

At the end of $\alpha=\pi/2$, there is a free moment condition

$$\frac{d\theta\left(\pi/2\right)}{d\alpha} = 1. \tag{43}$$

For later use, assume at $\alpha=\pi/2$ the deformed angle

$$\theta\left(\frac{\pi}{2}\right) = \gamma. \tag{44}$$

Substituting Eq. (43) into Eq. (41), the equation is then

$$\frac{1}{2}\left(\frac{d\theta}{d\alpha}\right)^2 = \frac{1}{2} + \lambda^2(\cos\theta - \cos\gamma). \tag{45}$$

Equation (45) can be suitably rearranged and integrated once with the boundary Eq. (42)

$$\int_0^{\gamma} \frac{d\theta}{\sqrt{\left[1 + 2\lambda^2(1 - \cos\gamma)\right] - 4\lambda^2 \sin^2 \theta/2}} = \frac{\pi}{2}. \tag{46}$$

Define

$$p^2 = \frac{4\lambda^2}{1 + 2\lambda^2(1 - \cos\gamma)} \qquad (0 < p^2 < 1). \tag{47}$$

The integral of Eq. (46) can be written as

$$\int_0^{\gamma/2} \frac{d\phi}{\sqrt{1 - p^2 \sin^2\phi}} = \frac{\lambda\pi}{2p}. \tag{48}$$

Hence the integral is an elliptic function of first kind, or

$$F\left(p, \frac{\gamma}{2}\right) = \frac{\lambda\pi}{2p}. \tag{49}$$

Eq. (49) can also be written as

$$F\left(\sqrt{\frac{4\lambda^2}{1+2\lambda^2(1-\cos\gamma)}},\frac{\gamma}{2}\right)= \frac{\pi}{4}\sqrt{1+2\lambda^2(1-\cos\gamma)} \tag{50}$$

Which are two variables of λ and γ . Once the force is applied, the deformed angle γ can be found. The various applied force versus deformed angle is shown in Fig. 6. Once the deformed angle at boundary is found, the horizontal coordinate at boundary x_a can be evaluated by integrating Eq. (14a) through half portion,

$$x_a=\int_0^{x_a}dx=\frac{p}{2\lambda}\int_0^{\gamma}\frac{\cos\theta}{\sqrt{1-p^2\sin^2\theta/2}}d\theta. \tag{51}$$

By similar procedure, the deformed coordinate at boundary

$$x_a=\frac{p}{\lambda}\left[\left(1-\frac{2}{p^2}\right)F\left(p,\frac{\gamma}{2}\right)+\frac{2}{p^2}E\left(p,\frac{\gamma}{2}\right)\right] \tag{52}$$

where *E(p, γ/2)* is the elliptic function of the second kind. The deformed vertical coordinate at *α=0* can be evaluating by

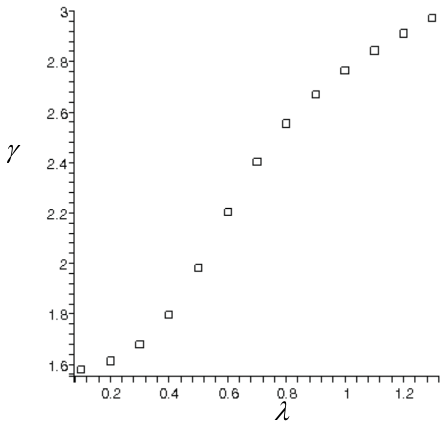

Fig. 6. The various applied force versus deformed angle.

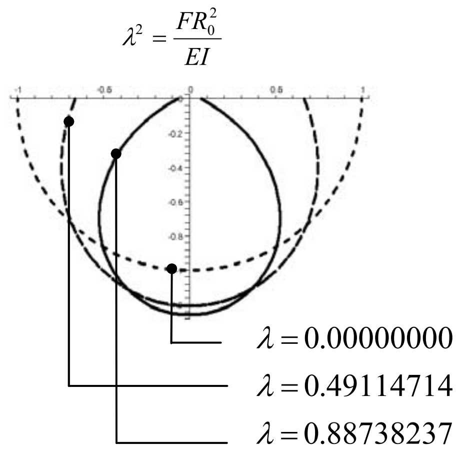

Fig. 7. The deformed shapes under loading of simple curved beams.

integrating Eq. (14b) as

$$y_a=1-\frac{1}{p\lambda}\left(1-\sqrt{1-p^2\sin^2\frac{\gamma}{2}}\right). \tag{53}$$

Note that the deformed angle γ is always greater than the deformed angle at any point, therefore *π/2>γ/2>θ/2*. This implies that the denominator of Eq. (50) is always greater than zero and is integrable. The deformed shapes under loading are shown in Fig. 7.

5. Conclusions

This paper has presented an analytical method for obtaining the finite deformation of 2-D circular curved beam. The rod axis is either inextensible or extensible. The curved beam is slender in the sense that dimension of cross section is much lessthan the dimension of radius of curvature. To derive the analytical method for the general solutions, one can introduce the coordinate system defined by the radius of centroidal axis and the angle of tangent slope.

The general solutions expressed by fundamental geometric quantities form a set of equations having seven unknown constants. The seven constants can be directly determined by suitable boundary conditions. As the radius in terms of the tangent slope angle is given, the fundamental geometric quantities can be calculated to obtain the closed form solutions of the axial force, shear force, bending moment, rotationangle, and deformed displacement fields at any cross-section of curved beams. These results of the applications indicate that the closed-form general solutions derived by the analytical method would be valid for in-plane thin curved beams. Thus the analytical method would be useful to engineers attempting to obtain the exact expressions for thin curved beams in engineering applications, especially, helical spring production and process.

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